Flexible Life Annuities

A. Direr*†

June, 4th 2008

Abstract

Annuity contracts typically deliver a stream of income at a predetermined level in order to insure against the risk of longevity. This paper explores whether flexible annuities, which give subscribers the possibility to choose between different levels of annuity, are welfare enhancing. In the case where agents gradually discover their actual probability of survival, a predetermined and "one-size-fits-all" annuity plan is optimal. If an expenditure risk is added along with the longevity risk, a flexible annuity plan is better even though the consumption path cannot be isolated from uninsured expenses anymore.

J.E.L. codes : E21, D82.

*Paris School of Economics (INRA), University of Grenoble. Address : 48 Bd Jourdan 75014 Paris. Email : direr@ens.fr

†I thank the participants of the Paris School of Economics "Health and Insurance" workshop and the CESifo Venice Summer Institute Workshop on Longevity and Annuitization. Any errors or omissions are my own.

1 Introduction

It is a well established fact that very few assets are converted into life annuities outside of social security and traditional defined-benefit pension plans. In the United States, for example, 401(k) pension plans are now the dominant form of private pension. Yet very few 401(k) plans offer the option to annuitize. More than 50% of households do not expect to even partially convert their defined-contribution account balances into an annuity (Brown, 2001). The United Kingdom has a long history of mandatory annuitization during retirement. In particular, defined-contribution pension funds are currently required to provide 75-year-old retirees with 75 percent of pension assets in the form of annuities. However, increasingly few insurers are willing to offer such products. In France, a voluntary and fully funded personal savings plan called PERP has been recently launched wherein buyers have to fully annuitize the wealth accumulated in their account at the date of retirement. So far, this constraint looks to have hindered its commercial development.

Underinvestment in life annuities may appear puzzling when considering the benefits that such annuities are expected to deliver to retirees. Yaari (1965) shows that full annuitization of assets is optimal in a standard model of saving without a bequest motive. Davidoff, Brown, and Diamond (2005) find an identical result for a large set of preferences and environments. A central question is hence why the annuity market is so small. An initial answer is that even though life annuities provide unequalled insurance against longevity, they also have particular disadvantages. Most obviously, the purchaser loses control over his assets, as most annuity plans deliver a predetermined lifelong income. Hence, once annuitized, wealth cannot serve to absorb unexpected income shocks, whereas most such income shocks for retirees are health-related. The risk of facing lower consumption following a period of unexpected health spending is seen as undermining demand for annuities (Brown, 2004). It has also a significant impact on saving behavior. For instance, a number of articles argue that the existence of an out-of-pocket medical expense risk explains why the elderly run down their assets so slowly (Palumbo 1999, De Nardi *et al.* 2006)

At first glance, this disadvantage could be minimized by giving the annuitant some flexibility over the annuity profile. A flexible contract can be defined as a contract that gives the annuitant the possibility of withdrawing a higher annuity in the event of an expense shock. The absence of such an option in existing contracts is generally justified by an argument of adverse selection as pointed up by Brown and Warshawsky (2001, p.14): "Insurance companies do not allow individuals to cancel an annuity agreement once it is in place. Otherwise adverse selection would obviously occur as individuals acquire information about their expected longevity." If individuals were to update their actual chances of longevity, annuitants expecting to live longer would indeed be likely to increase their financial stake in the plan while the shorter-lived consumers would more likely opt to consume larger amounts earlier, thereby reducing their annuitized wealth. If the insurer were to give subscribers the flexibility to choose between different levels of annuity, the expected return on a pool of savings could be greatly reduced by an adverse shift in the composition of the population of subscribers. At the outside, if agents were always informed well enough in advance of the day on which they would die, they would have the time to close their personal account and escape the redistribution scheme. This would cause the collapse of the longevity insurance mechanism.

Interestingly, insurance companies have started introducing a number of new annuity products, which soften the liquidity constraint (Brown, 2007). Nearly 80 percent of the variable annuities sold in the U.S. in the first quarter of 2006 included a guaranteed minimum withdrawal benefit which guarantees a minimum level of income with the annuitants being able to adjust the withdrawal amount. Brown (2007) also mentions an insurance company that offers a fixed life annuity with an option to withdraw, on a one-time only basis, up to 30 percent of the expected value of the remaining annuity payments based on mortality rates at the time of purchase. This option can only be exercised on the 5th, 10th, or 15th anniversary of the first payment.

The objective of this paper is to ask whether adverse selection dismisses the feasibility of flexibility in annuity plans. I begin by presenting a simple model in which life expectancy is the only source of uncertainty. Agents update their mortality risk during the retirement phase. This raises the question as to whether they should adjust the level of their annuity based on their more accurate estimate of mortality. This simple model shows that they should not. A fixed annuity plan which provides a predetermined stream of annuities independent of future information about longevity, is optimal because it prevents annuitants from drawing their wealth down in the event of bad news about their mortality. This is a clear case in which flexible life annuities are not welfare enhancing, in line with the argument put forward by Brown and Warshawsky (2001).

The model is subsequently extended by including a liquidity risk in addition to the longevity risk. Some savers face a liquidity need due to adverse health shocks. Some flexibility in the form of a choice between different levels of annuities then becomes optimal. However, the risk of increased expenses cannot be fully insured due to the abovementioned adverse selection mechanism.

The main finding of the analysis is that adverse selection *per se* does not justify the complete absence of flexibility. The option of raising an annuity during a given period up to a certain limit can cover some additional expenses and thereby provide insurance. Flexibility is then optimal, even though the rate of return is reduced due to adverse selection.

The issue addressed by this article is formally close to Brugiavini (1993). She studies at which stage of their life individuals who learn about their survival probability as they get older, should purchase annuities. Her model predicts a purchase in an early stage of the life cycle. The present model finds similar implications if the period during which individuals update their survival probability is reinterpreted as a retirement period instead of an early working period. However, the two papers address different questions. Her paper asks whether workers should buy an annuity contract at an early stage of their life and, if they do, whether they could recontract when they retire. The present paper studies whether it is optimal to propose several levels of annuities to retirees. A formal connection between the two setups is explored further at the end of Section 2.

The plan is structured as follows. Section 2 presents a simple model of annuity demand under uncertain longevity and puts forward a basic argument as to why a fixed annuity plan may be optimal. Section 3 introduces a liquidity shock and studies the extent to which this additional source of uncertainty may be optimally insured by a flexible annuity plan. The last section concludes.

2 A model with uncertainty about longevity

This section presents a formal case against flexible annuities in a basic set-up in which the only uncertainty is a longevity risk. This case will serve as a benchmark model when a second source of uncertainty is introduced in the next section.

Consider an environment in which agents allocate their wealth w between consumption on two dates $t = 1, 2$. The gross interest rate of the economy is denoted R. Life expectancy is uncertain such that individuals face a probability of dying before their last period of consumption. The uncertainty of survival calls for the purchase of annuities, which pay the living a premium in return for the subscribers' wealth upon death. In the absence of a bequest motive or uninsurable risk, full annuitization prevails (Yaari, 1965).

In the model, uncertainty about the mortality risk gradually resolves itself between an initial date 0 and date 1. Individuals have the same life expectancy at $t = 0$ and are all alive at date 1. At $t = 1$, they obtain more accurate information about their actual probability of surviving the last period $t = 2$. A fraction p of annuitants learn that their probability of survival is π_h. The remaining annuitants are characterized by a lower survival probability $\pi_l < \pi_h$. c_{ti} denotes consumption at date $t = 1, 2$ of agents with longevity type $i = h, l$. Utility of consumption is denoted $u(c)$, with $u'(c) > 0$, $u''(c) < 0$ and $\lim u'(c) = \infty$ when $c \to 0$. Intertemporal utility is additive. β is the subjective discount rate attached to utility of last period.

$$v(c_{1i}, c_{2i}; \pi_i) = u(c_{1i}) + \pi_i \beta u(c_{2i})$$

Information about longevity is private. Hence, insurers are unable to separate out annuitants by risk classes. Moreover they cannot monitor whether costumers hold annuities from other firms. They compete by offering the most attractive rate of return for savings. Individuals can purchase as many annuities as they want at the prevailing rate of return. This leads to the definition of an asymmetric information equilibrium (e.g. Abel, 1986) characterized by the annuity plan $\{(\widehat{c}_{1i}, \widehat{c}_{2i}); i = h, l\}$ which satisfies:

$$\begin{cases} (\widehat{c}_{1i}, \widehat{c}_{2i}) = \arg \begin{cases} \max v(c_{1i}, c_{2i}; \pi_i) \\ \text{s.t. } c_{1i} + \overline{\pi} c_{2i}/R = w \end{cases} & i = h, l \\ \overline{\pi} = \frac{p\widehat{c}_{2h}}{p\widehat{c}_{2h} + (1-p)\widehat{c}_{2l}} \pi_h + \frac{(1-p)\widehat{c}_{2l}}{p\widehat{c}_{2h} + (1-p)\widehat{c}_{2l}} \pi_l \end{cases} \tag{1}$$

Insurers provide annuitants with the highest actuarially feasible rate of return $R/\overline{\pi}$. This rate takes as given the average survival rate $\overline{\pi}$, weighted by the participation rate of each type in the annuity plan. Given this rate of return, consumers choose the best annuity profile according to the above definition.

Now, assume that consumers have the possibility of signing a binding contract at date 0 at a time when they have not yet updated their longevity information. Insurers collect savings and offer an annuity contract that maximizes expected utility at date 0:

$$\begin{cases} \max pv(c_{1h}, c_{2h}; \pi_h) + (1-p)v(c_{1l}, c_{2l}; \pi_l) \\ \text{s.t. } p(w - c_{1h} - \pi_h c_{2h}/R) + (1-p)(w - c_{1l} - \pi_l c_{2l}/R) = 0 \end{cases} \tag{2}$$

The resulting consumption path $\{(c_{1i}^*, c_{2i}^*); i = h, l\}$ is defined by:

$$\begin{cases} c_{th}^* = c_{tl}^* & t = 1, 2 \\ u'(c_{1i}) = \beta R u'(c_{2i}) & i = 1, 2 \end{cases} \tag{3}$$

The allocation is Pareto optimal since information is symmetric at date 0. It equalizes consumption across risks and makes the consumption path independent of the mortality risk.

In this equilibrium, insurers compete for savings as early as date 0 by promising an annuity sequence $\{(c^*_{1i}, c^*_{2i}); i = h, l\}$. It remains to be seen whether individuals are willing to subscribe to such a contract or whether they prefer to postpone their consumption decision. Indeed, even though this contract is optimal from an ex-ante point of view, it is obviously not a contract that retirees would buy at date 1 once their individual longevity is better assessed.

Proposition 1. Date 0 expected utility from subscribing to the ex-ante optimal contract is higher than the expected utility derived from postponing the consumption decision until date 1 (proof in appendix).

There is a basic intuition behind this result, first outlined by Hirshleifer (1971). The reduction of uncertainty at date 1 prevents ex ante optimal transfers from low risk agents to high risk agents, which undermines the insurance scheme. Proposition 1 associated with Eq. (3) present a clear case against flexible life annuities since they establish the superiority of an unconditional annuity plan that delivers a single level of annuity at dates 1 and 2. By prohibiting agents from receiving a higher annuity in the event of a negative signal about their longevity, a fixed annuity plan preserves a higher rate of return, and means that the risk of longevity is mutualized more efficiently.

Although it is optimal to buy a predetermined annuity plan at date 0, it remains to be seen whether agents cannot improve their situation by recontracting at date 1, that is by buying or selling new annuities (proof in appendix).

Proposition 2. Agents do not recontract at date 1.

This sums up the argument put forward by Brugiavini (1993). Agents do not recontract later since the effect of new longevity information on consumption is exactlyoffset by the revision of the actuarially fair interest rate.

This section has shown the optimal nature of a fixed annuity plan. The next section adds in uninsurable expenses, which produces a different result.

3 Adding uninsurable expense shocks

The above framework is the same in all respects except that some uninsurable expenses, possibly health related, may be incurred at date 1. There are now three sorts of consumers who discover their type at date 1. The first, in proportion p, is a long-lived consumer who survives date 2 with probability $\pi > 0$. Such agents are denoted as being of type h below. Agents of a second type m are represented in proportion q and also survive with probability π, but incur an additional cost $m > 0$ at date 1. This cost is interpreted as an out-of-pocket medical expense. Lastly, type l agents, in proportion $s = 1 - p - q$, are short-lived consumers who die with certainty by the end of period 1. The assumption that short-lived agents do not survive with a positive probability is not essential to this section's argument and simplifies the analysis.

In the case where agents buy an annuity contract at date 1, only individuals with a positive survival probability will want to buy further annuities. Insurers will therefore sell annuities with a rate of return R/π. However, this contract is not optimal from an ex ante point of view. Accordingly, the rest of this section will concentrate on the optimal annuity contract at date 0, at a time when agents do not know their type and some insurance is still possible.

An optimal plan is a set of annuities $\{(c_{1h}, c_{2h});\ (c_{1m}, c_{2m});\ (c_{1l}, 0)\}$ dedicated to types h, m and l respectively. It is assumed that agents are able to partially recontract in date 1. They may buy more annuities in order to consume more at the last date. They are however prevented from short-selling annuities and therefore cannot consume less than c_{2i} $(i = h, m)$.

It is informative to start the analysis with the hypothesis of complete information. Insurance companies offer a contract at date 0 that specifies annuity payouts for period 1 and 2 which differ according to the agents' actual situation (or type). Agents ignore their type but rationally buy a contract which encompasses three different annuity options. At date 1, each agent's type is known to everyone. Insurance companies provide agents with the corresponding annuity option that was specified in the contract. In this scenario, annuity contracts can be tailored to each type and full insurance between the insured prevails: $c_{1h}^* = c_{1m}^* - m = c_{1l}^*$ and $c_{2h}^* = c_{2m}^*$. The risk of longevity and the expense shock are perfectly mutualized across agents. Moreover, consumption evolves in accordance with the first best rule: $u'(c_{1i}^*) = \beta R u'(c_{2i}^*)$, $i = h, l$.

In the more realistic case in which an agent's type is private information, the insurer has to make sure that each annuity plan is chosen by the type for whom it is designed. A first constraint checks that type h agents actually choose the annuity plan (c_{1h}, c_{2h}) instead of (c_{1m}, c_{2m}) designed for type m individuals. This happens if type h agents do not achieve a higher utility by selecting c_{1m}, buying the additional amount of annuities $x = c_{1m} - c_{1h}$ and then consuming $c_{2h} = c_{2m} + xR/\pi$ in the last period. The possibility of switching to a different annuity profile leads to a reservation level of utility denoted by v_h:

$$v_h(z) = \{\max u(c_1) + \pi\beta u(c_2); c_1 + c_2\pi/R = z\} \tag{4}$$

where $z = c_{1m} + \pi c_{2m}/R$ is the intertemporal value of the annuity plan (c_{1m}, c_{2m}). The incentive compatibility constraint checks that:

$$u(c_{1h}) + \pi\beta u(c_{2h}) \geq v_h(z) \quad (5)$$

Note that a first best policy with full information redistributes resources to the agents who incur the expense shock because $c_{1h} = c_{1m} - m < c_{1m}$ and $c_{2h} = c_{2m}$. When the type is private information, the insurer is constrained to give the same actuarial value to both types (see the resource constraint in (4)). For the same reason, the reverse constraint in which type m would prefer (c_{1h}, c_{2h}) to (c_{1m}, c_{2m}) is not operative in equilibrium as the insurance scheme implies transferring more resources to type m, not less.

A second incentive compatibility constraint recognizes that type l agents always choose the highest annuity available at date 1. It follows that the annuity c_{1l} cannot be less than the same period annuity designed for long-lived agents: $c_{1l} \geq \max(c_{1h}, c_{1m})$.

An equilibrium set of contracts maximizes the consumers' expected utility given a global resource constraint and the two information constraints:

$$\left\{ \begin{array}{ll} \max p(u(c_{1h}) + \pi\beta u(c_{2h})) + q(u(c_{1m} - m) + \pi\beta u(c_{2m})) + su(c_{1l}) & \\ & p(c_{1h} + \pi c_{2h}/R) + q(c_{1m} + \pi c_{2m}/R) + sc_{1l} = w \\ \text{s.t.} & u(c_{1h}) + \pi\beta u(c_{2h}) \geq v_h(z) \\ & c_{1l} \geq \max(c_{1h}, c_{1m}) \end{array} \right. \quad (6)$$

It is straightforward that $c_{1m} \geq c_{1h}$ in equilibrium. If this were not the case, the insurer could marginally reduce the date 1 consumption of type h and increase the consumption of type m by a factor p/q, yielding a net utility increase of $u'(c_{1m} - m) - u'(c_{1h}) > 0$. Hence, it follows that the latter information constraint simplifies to

$$c_{1l} \geq c_{1m} \tag{7}$$

The program (6) can be rewritten in a more convenient way:

$$\begin{cases} \max p(u(c_{1h}(z)) + \pi\beta u(c_{2h}(z))) + q(u(c_{1m} - m) + \pi\beta u(c_{2m})) + su(c_{1l}) \\ \text{s.t.} \quad (1-s)(c_{1m} + \pi c_{2m}/R) + sc_{1l} = w \\ c_{1l} \geq c_{1m} \end{cases}$$

where $c_{1h}(z)$ and $c_{2h}(z)$ are agents' optimal demands for annuities solving (4). The second constraint merges with the budget constraint of (4) into the broader budget constraint of the insurer in (6). We can next examine the consequences for the optimal annuity plan.

Lemma 1. The short-lived agents and the agents incurring an expense shock are given the same level of annuity at date 1: $c_{1l} = c_{1m}$.

The basic reason behind this equality is that an expense shock is a legitimate reason for allowing the saver to withdraw a higher annuity, whereas a negative signal about survival is not. As a result, the insurer would prefer to offer a smaller annuity to type l compared to the one proposed to type m. As it is impossible to distinguish between the two types, the insurer is bound to offer a single annuity level to both types.

In other words, a high annuity is required for type m agents to ease the expense shock. At the same time, a high annuity benefits the short-lived agents and magnifies the adverse selection effect. It follows that insurers cannot simultaneously and perfectly insure against longevity and expense shocks. This contradiction is reflected in Lemma 2.

Lemma 2. Annuitants who experience an expense shock cannot withdraw a high enough annuity to fully smooth their consumption profile compared to the first best environment, that is: $u'(c_{1m} - m) > R\beta u'(c_{2m})$.

The lack of insurance is entirely due to the presence of short-lived agents, as confirmed by Lemma 3:

Lemma 3. Perfect consumption smoothing could be restored if the adverse selection effect were absent. Formally, if $s = 0$, $u'(c_{1m} - m) = R\beta u'(c_{2m})$.

Finally, the question may be asked as to whether the annuitants benefit at all from an insurance mechanism. In other words, is it optimal to propose an annuity plan that includes a high annuity level that can be selected in the event of an expense shock ?

Proposition 3. A flexible plan is optimal in the form of a withdrawal option. The agents incurring an expense shock benefit from a higher annuity than the long-lived agents without the expense shock : $c_{1m} > c_{1h}$.

This proposition states that consumers have the option of withdrawing $c_{1m} - c_{1h}$, leading to an annuity reduction $c_{2m} - c_{2h}$ in period 2. As regards the question raised in the introduction, this means that a flexible plan is now optimal.

To sum up, the insurer proposes three annuity plans (c_{1h}, c_{2h}), (c_{1m}, c_{2m}) and $(c_{1l}, 0)\}$, one for each type. A certain degree of flexibility is optimal, as agents are free to choose between consuming a high annuity c_{1m} and a low one c_{1h}. Choosing the high annuity reduces the last-period income but improves the well-being of the agents faced with an expense shock at date 1. Short-lived agents choose to "close" their annuity plan by consuming the high level $c_{1l} > c_{1h}$ as well. However, they face a restriction, as they cannot consume more than the level of annuity that agents with an expense shock are allowed to withdraw. This restriction is optimal as it limits the scope of the adverse selection effect. As insurers do not allow short-lived agents to deplete their account at date 1, agents in need of additional income cannot fully smooth their consumption profile.

4 Conclusion

This paper studies the optimal annuity contract for agents faced with a liquidity risk and a longevity risk. A simultaneous analysis of the effects of both types of risk is relevant. Taken separately, they produce conflicting predictions as regards contract characteristics. With both types of risk, flexible annuities whereby annuitants can choose between different withdrawal levels are welfare-enhancing as they allow agents suffering an expense shock to better smooth their consumption path.

It is worth noting that flexibility does not mean total freedom for annuitants. Indeed, the high pay-out level cannot be raised too much, as it leads the short-lived individuals to deplete their saving account, thereby undermining the longevity insurance scheme. This constraint prevents agents who experience an expense shock from perfectly smoothing their consumption profile.

The paper's main finding is that the adverse selection effect sometimes stated as an argument against flexible annuities is not sufficient for the optimality of a fixed and "one-size-fits-all" annuity scheme. The option to increase withdrawals in a given period up to a limited level may achieve a good deal of consumption smoothing while limiting the size of the rate-of-return reduction due to the adverse selection effect. A practical conclusion is that a minimal degree of flexibility could well promote wealth annuitization by reducing the mismatch between the desired consumption path and the annuity income stream.

Flexibility could also entail additional administrative costs, which would make this option more expensive for retirees than the model assumes. How administrative costs vary with the level of annuity contract sophistication is primarily an empirical issue, which would call for further exploration.

References

Abel A. B. (1986) "Capital Accumulation and Uncertain Lifetimes with Adverse Selection" *Econometrica*, 54 (5) 1079-1097.

Brown J. R. (2001) "Private Pensions, Mortality Risk, and the Decision to Annuitize" *Journal of Public Economics* 82 (1) 29-62.

Brown, J. R. (2004) “Life Annuities and Uncertain Lifetimes", NBER Reporter: Research Summary Spring.

Brown, J. R. (2007) “Rational and Behavioral Perspectives on the Role of Annuities in Retirement Planning", NBER Working Paper 13537.

Brown, J. R., and M. J. Warshawsky (2001) "Longevity-Insured Retirement Distributions from Pension Plans: Market and Regulatory Issues", NBER Working Paper 8064.

Brugiavini, A. (1993) "Uncertainty Resolution and the Timing of Annuity Purchases", *Journal of Public Economics* 50 (1): 31-62.

De Nardi M., French E., and J. B. John (2006) "Differential Mortality, Uncertain Medical Expenses, and the saving of Ederly Singles" NBER Working Paper 12554.

Hirshleifer, J. (1971) "The Private and Social Value of Information and the Reward to Inventive Activity", *American Economic Review*, volume 61, no. 4, 561–574.

Palumbo M. G. (1999) "Uncertain Medical Expenses and Precautionary Saving Near the End of the Life Cycle" *Review of Economic Studies*, 66 395-421.

Yaari, M.E. (1965) "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer", *Review of Economic Studies* 32, 2, 137-150.

Appendix

Proof of Proposition 1. We set out to show that :

$$pv(c^*_{1h}, c^*_{2h}; \pi_h) + (1-p)v(c^*_{1l}, c^*_{2l}; \pi_l) > pv(\widehat{c}_{1h}, \widehat{c}_{2h}; \pi_h) + (1-p)v(\widehat{c}_{1l}, \widehat{c}_{2l}; \pi_l).$$

By way of comparison, the date 0 problem in (2) can be recast into:

$$\left\{ \begin{array}{c} \max pv(c_{1h}, c_{2h}; \pi_h) + (1-p)v(c_{1l}, c_{2l}; \pi_l) \\ \text{s.t. } p(w - c_{1h} - \overline{\pi}c_{2h}/R) + (1-p)(w - c_{1l} - \overline{\pi}c_{2l}/R) = 0 \\ \overline{\pi} = \frac{pc_{2h}}{pc_{2h}+(1-p)c_{2l}}\pi_h + \frac{(1-p)c_{2l}}{pc_{2h}+(1-p)c_{2l}}\pi_l \end{array} \right.$$

This program closely resembles the ex-post problem with asymmetric information defined in (1):

$$\begin{array}{llll} (i) & \{(\widehat{c}_{1i}, \widehat{c}_{2i}); i = h, l\} & = & \arg \left\{ \begin{array}{c} \max pv(c_{1h}, c_{2h}; \pi_h) + (1-p)v(c_{1l}, c_{2l}; \pi_l) \\ \text{s.t. } c_{1i} + \overline{\pi}c_{2i}/R = w \qquad i = h, l \end{array} \right\} \\ (ii) & \overline{\pi} & = & \frac{p\widehat{c}_{2h}}{p\widehat{c}_{2h} + (1-p)\widehat{c}_{2l}}\pi_h + \frac{(1-p)\widehat{c}_{2l}}{p\widehat{c}_{2h} + (1-p)\widehat{c}_{2l}}\pi_l \end{array}$$

except that the budget constraint is now split into two separate constraints instead of one and that the impact of the consumption choice on the rate of return $R/\overline{\pi}$ is not internalized. Hence, despite the objective being the same, the ex-post problem includes additional constraints. Moreover, at least one of these constraints must be binding since the ex-post problem implies $\widehat{c}_{1h} < \widehat{c}_{1l}$ whereas the ex-ante problem leads to $c^*_{1h} = c^*_{1l}$. As a result, it delivers a less efficient consumption stream from the point of view of date 0.

Proof of proposition 2. Let us denote by b_i the additional amount of annuities that agents of type i may buy (or sell short) at date 1 for the last period of consumption. Since they have invested all their savings in the annuity plan at date 0, they use the payment c^*_1 to consume and purchase additional annuities at date 1: $\widetilde{c}_{1i} + b_i = c^*_1$ such that their date 2 consumption if alive is $\widetilde{c}_{2i} = c^*_2 + Rb_i/\pi_i$. The rate of return is adjusted to each specific risk, since buying new annuities reveals a high risk and short-selling reveals a low risk. Hence, the revised consumption plan is determined by:

$$\left\{ \begin{array}{c} \max v(c_{1i}, c_{2i}; \pi_i) \\ \text{s.t. } c_{1i} + \pi_i c_{2i}/R = c^*_1 + \pi_i c^*_2/R \end{array} \right\} \qquad i = h, l$$

This plan does not deviate from the ex ante optimal allocation, which means that $b_i = 0$.

Proof of Lemma 1.

We set out to show that the incentive compatibility constraint (7) is binding:

$c_{1l} = c_{1m}$. If $c_{1l} \neq c_{1m}$, the insurer's program is:

$$\begin{cases} \max p(u(c_{1h}(z)) + \pi\beta u(c_{2h}(z))) + q(u(c_{1m} - m) + \pi\beta u(c_{2m})) + su(c_{1l}) \\ \text{s.t.} \quad (p+q)(c_{1m} + \pi c_{2m}/R) + sc_{1l} = w \end{cases}$$

Forming the Lagrangian function:

$$\begin{aligned} L(c_{1m}, c_{2m}, c_{1l}) &= p(u(c_{1h}(z)) + \pi\beta u(c_{2h}(z))) + q(u(c_{1m} - m) + \pi\beta u(c_{2m})) + su(c_{1l}) \\ &\quad + \lambda[w - (p+q)(c_{1m} + \pi c_{2m}/R) - sc_{1l}] \end{aligned}$$

and setting the partial derivatives to zero with respect to c_{1m} and c_{1l}:

$$\begin{aligned} \frac{\partial L}{\partial c_{1m}} &= p[u'(c_{1h})\frac{\partial c_{1h}}{\partial z} + \pi\beta u'(c_{2h})\frac{\partial c_{2h}}{\partial z}] + qu'(c_{1m} - m) - (p+q)\lambda = 0 \\ \frac{\partial L}{\partial c_{1l}} &= su'(c_{1l}) - s\lambda = 0 \end{aligned}$$

The following inequalities can then be derived:

$$\begin{aligned} u'(c_{1l}) &= \frac{p}{p+q}[u'(c_{1h})\frac{\partial c_{1h}}{\partial z} + \pi\beta u'(c_{2h})\frac{\partial c_{2h}}{\partial z}] + \frac{q}{p+q}u'(c_{1m} - m) \\ &= \frac{p}{p+q}u'(c_{1h})[\frac{\partial c_{1h}}{\partial z} + \frac{\pi}{R}\frac{\partial c_{2h}}{\partial z}] + \frac{q}{p+q}u'(c_{1m} - m) \qquad \text{(i)} \\ &= \frac{p}{p+q}u'(c_{1h}) + \frac{q}{p+q}u'(c_{1m} - m) \qquad \text{(ii)} \\ &\geq \frac{p}{p+q}u'(c_{1m}) + \frac{q}{p+q}u'(c_{1m}) = u'(c_{1m}) \qquad \text{(iii)} \end{aligned}$$

Equality (i) is obtained by using the Euler equation $u'(c_{1h}) = \beta Ru'(c_{2h})$ derived from (4), (ii) by differentiating the budget constraint (4): $c_{1h}(z) + c_{2h}(z)\pi/R = z$ and substituting for $\partial c_{1h}/\partial z$. Inequality (iii) exploits the fact that $c_{1m} \geq c_{1h}$ at equilibrium. The result is that $c_{1l} \leq c_{1m}$ or $c_{1l} = c_{1m}$ when the information constraint (7) is taken into account.

Proof of Lemma 2.

The insurer's program with $c_{1l} = c_{1m}$ is:

$$\begin{cases} \max p(u(c_{1h}(z)) + \pi\beta u(c_{2h}(z))) + q(u(c_{1m} - m) + \pi\beta u(c_{2m})) + su(c_{1m}) \\ \text{s.t.} \; c_{1m} + (1-s)c_{2m}\pi/R = w \end{cases}$$

Forming the Lagrangian function and setting the partial derivatives to zero:

$$\begin{aligned}
\frac{\partial L}{\partial c_{1m}} &= pu'(c_{1h})\frac{\partial c_{1h}}{\partial z} + p\pi\beta u'(c_{2h})\frac{\partial c_{2h}}{\partial z} + qu'(c_{1m} - m) + su'(c_{1m}) - \lambda = 0 \\
\frac{\partial L}{\partial c_{2m}} &= pu'(c_{1h})\frac{\partial c_{1h}}{\partial z}\frac{\pi}{R} + p\pi\beta u'(c_{2h})\frac{\partial c_{2h}}{\partial z}\frac{\pi}{R} + q\pi\beta u'(c_{2m}) - (1-s)\frac{\pi}{R}\lambda = 0
\end{aligned}$$

Substituting for the multiplier λ:

$$\begin{aligned}
& \frac{p}{1-s}u'(c_{1h})\frac{\partial c_{1h}}{\partial z} + \frac{p}{1-s}\pi\beta u'(c_{2h})\frac{\partial c_{2h}}{\partial z} + \frac{q}{1-s}\beta R u'(c_{2m}) \\
= \; & pu'(c_{1h})\frac{\partial c_{1h}}{\partial z} + p\pi\beta u'(c_{2h})\frac{\partial c_{2h}}{\partial z} + qu'(c_{1m} - m) + su'(c_{1m})
\end{aligned}$$

Using $u'(c_{1h}) = R\beta u'(c_{2h})$:

$$\begin{aligned}
\frac{q}{1-s}R\beta u'(c_{2m}) &= qu'(c_{1m} - m) + su'(c_{1m}) \\
+ & \quad u'(c_{1h})\left[p\frac{\partial c_{1h}}{\partial z} + p\frac{\pi}{R}\frac{\partial c_{2h}}{\partial z} - \frac{p}{1-s}\frac{\partial c_{1h}}{\partial z} - \frac{p}{1-s}\frac{\pi}{R}\frac{\partial c_{2h}}{\partial z}\right]
\end{aligned}$$

Differentiating $c_{1h}(z) + c_{2h}(z)\pi/R = z$ and substituting for $\partial c_{1h}/\partial z$ leads to the constrained Euler equation:

$$\begin{aligned}
R\beta u'(c_{2m}) &= (1-s)u'(c_{1m} - m) + \frac{s(1-s)}{q}u'(c_{1m}) - \frac{ps}{q}u'(c_{1h}) && (8) \\
&< (1-s)u'(c_{1m} - m) + \frac{s(1-s)}{q}u'(c_{1h}) - \frac{ps}{q}u'(c_{1h}) && \text{(i)} \\
&= (1-s)u'(c_{1m} - m) + su'(c_{1h}) \\
&\leq (1-s)u'(c_{1m} - m) + su'(c_{1m} - m) && \text{(ii)} \\
&= u'(c_{1m} - m)
\end{aligned}$$

Inequality (i) draws on the fact that $c_{1m} > c_{1h}$ and (ii) $c_{1m} - m \leq c_{1h}$. Indeed, $c_{1m} - m > c_{1h}$ would lead to over-insurance. In this case, $u'(c_{1m} - m) < u'(c_{1h}) = R\beta u'(c_{2h})$. Moreover, $c_{1m} > c_{1h}$ implies $c_{2m} < c_{2h}$ from (4) and therefore $R\beta u'(c_{2m}) > R\beta u'(c_{2h})$ implying $u'(c_{1m} - m) < R\beta u'(c_{2m})$. That is, type m agents consume too much at date 1 despite additional expenses. It follows that insurers could improve the position of type m by reducing c_{1m} and raising c_{2m} without violating the type m resource constraint $c_{1m} + \pi c_{2m}/R = w$ and the incentive compatibility constraints.

Proof of Lemma 3.

Take the constrained Euler equation (8) found in the previous proof and set $s = 0$ and $p + q = 1$.

Proof of Proposition 3.

Let us prove that $c_{1m} = c_{1h}$ cannot hold at equilibrium. This implies $c_{2m} = c_{2h}$ from (4). The insurer's program with $c_{1h} = c_{1m} = c_{1l}$ is then:

$$\begin{cases} \max(1-q)u(c_{1m}) + qu(c_{1m} - m) + (1-s)\pi\beta u(c_{2m}) \\ \text{s.t. } c_{1m} + (1-s)c_{2m}\pi/R = w \end{cases}$$

Forming the Lagrangian function and setting the partial derivatives to zero lead to the Euler equation: $(1-q)u'(c_{1m}) + qu'(c_{1m} - m) = R\beta u'(c_{2m})$. Hence $u'(c_{1m}) < R\beta u'(c_{2m})$. But $c_{1m} = c_{1h}$ also implies $c_{2m} = c_{2h}$ and $u'(c_{1m}) = R\beta u'(c_{2m})$ from (4), which contradicts the former equation.