Optimal Lottery*

Charles Dennery — London School of Economics

Alexis Direr† — Université d'Orléans

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Abstract

This article proposes an equilibrium approach to lottery markets in which a firm designs an optimal lottery to rank-dependent expected utility (RDU) consumers. We show that a finite number of prizes cannot be optimal, unless implausible utility and probability weighting functions are assumed. We then investigate the conditions under which a probability density function can be optimal. With standard RDU preferences, this implies a discrete probability on the ticket price, and a continuous probability on prizes afterwards. Under some preferences consistent with experimental literature, the optimal lottery follows a power-law distribution, with a plausibly extremely high degree of prize skewness.

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†Univ. Orléans, CNRS, LEO, UMR 7322, F45067, and Paris School of Economics. E-mail : direr@ens.fr.

1 Introduction

The popularity of commercial lotteries offering big prizes with small probabilities reveals a demand for positively skewed lotteries. Skewness preferences arise if gamblers overweight the upper tail of probability distributions. They are commonly modeled by rank-dependent expected utility (Quiggin, 1982), a leading theory of choice under risk, in which agents transform cumulative probabilities.

This explanation provides an intuitive account of lottery demand but falls short as a comprehensive description of lottery markets as the supply side of lotteries is generally not considered. In particular, it remains unclear which exact form should take a lottery (minimal prize, number of tickets, degree of skewness, ...) compatible with operators maximizing their profit. An equilibrium approach to lottery market is also a first necessary step towards more applied or regulatory issues like tax efficiency and dead-weight loss of lottery games, consequences of legal monopolies, existence of scale economies, price and income elasticities of demand, optimal prize structure, to name a few (see Grote and Matheson (2011) for a recent survey on those issues).

This article aims at filling part of the gap by investigating two related issues: which preference patterns are compatible with profit-maximizing lotteries endowed with multiple prizes and which form take optimal lotteries when consumers are characterized by realistic Rank-Dependent Expected Utility (RDU) preferences? By allowing the lottery operator to freely choose prize values, their probability, and the number of prizes, we show that a finite number of prizes would require both the utility and the weighting function to turn concave then convex or vice-versa each time a new prize is added to the lottery. In a two-payoff lottery, the utility function has a concave-convex-concave shape as in Friedman and Savage (1948) who study the expected utility case. With more than two payoffs, the more realistic case, the number of alternations of the curvature of the two functions becomes implausible.

However, and this is the second part of our paper, RDU preferences naturally fit with continuous probability distributions. In particular, if the utility function is concave, and the weighting function has an inverse-S shape, as the empirical literature suggests, there will be a mass of probability on the worst outcome (paying the ticket price), and the probability distribution will be continuous afterwards. A fundamental characteristic of lottery games is also their very high degree of positive skewness with extremely large jackpots offered with close-to-zero probabilities. We show that prizes over the continuous part of the distribution optimally follow a power-law distribution when realistic functionals for the utility and the weighting functions are chosen. We illustrate our result with a calibration exercise which uses prize data from Euromillions, a Europe-wide lottery game. We document a very high degree of skewness and show its consistency with reasonably calibrated RDU preferences.

Our approach is in the spirit of Friedman and Savage (1948) who rationalize the demand for lottery tickets in the expected utility (EU) framework. Assuming an increasing marginal utility for a broad range of wealth provides a rationale for two-outcome lotteries, but accounts for very limited patterns of gambling. Markowitz (2010) shows that EU is unable to explain the existence of optimal lotteries with more than two payoffs. We extend his negative result in the more general RDU framework with an arbitrary -yet discrete- number of prizes. Quiggin (1991) also studies the optimal shape of a lottery in RDU and shows the possibility of lotteries with multiple prizes. His argument is however developed in a simplified setup with an exogenous number of equally probable tickets. Once the number of prizes and their probability are made endogenous, we show that a lottery operator has always the incentives to add new prizes between existing ones, provided implausible preference patterns are excluded. Hence the only equilibrium outcome is a continuous prize distribution. Barberis (2012) uses cumulative prospect theory, a variant of RDU, to explain the demand for casino gambling in an intertemporal setup. He does not endogenize the prize structure.

Our results are also related to the burgeoning literature which shows evidence or examine implications of a demand for skew. Garrett and Sobel (1999) and Astebro et al. (2011) show evidence that consumers favor skewness rather than risk in lottery games. Asterbro et al. experimentally explain skewness preferences by small probabilities overweighting (inverse S-shaped weighting function) rather than risk loving (convex utility function). Snowberg and Wolfers (2010) investigate the favourite-longshot bias in horse races betting and reach similar conclusions. Barberis (2013) mention several articles in which RDU models explain a demand for skewness. In financial markets Barberis and Huang (2008) show that probability weighting may explain why positively skewed securities are overpriced at equilibrium. We contribute to this literature by showing that probability weighting is a better candidate than risk loving to explain multiple-prize lotteries. The optimal degree of skewness is the result of two opposite psychological factors. On the one hand, a concave utility function makes more costly the spread of prize payouts and the inclusion of extreme payoffs. On the other hand a convex weighting function for cumulative probabilities close to one leads consumers to overweight small probabilities associated with extreme prizes, which strengthens the demand for skew. We also link probability weighting with a demand for power-law distributions which have attracted much attention (see e.g. Gabaix, 2011).

Last, our setting is broadly connected to the literature which studies risk sharing between non-expected-utility consumers. Chateauneuf et al. (2000) examine risk sharing arrangement between risk-averse Choquet expected utility agents in special cases. Dana and Carlier (2008) extend the analysis to a broad class of non-expected utility models. Bernard et al. (2013) analyze the optimal insurance contract problem between a risk neutral agent and a RDU agent with an inverse S-shaped probability weighting function. A fundamental difference between those articles and the present one is that consumption risk does not preexist in our environment. To fulfill the demand for risk taking by RDU agents, the lottery operator makes monetary transfers contingent to a "randomization device". In real world, this could be a rotating ball cage or scratch cards.

The presentation is organized as follows. Section 2 studies to what extent RDU preferences may explain the existence of optimal lotteries endowed with a discrete number of prizes. Section 3 reverses the perspective and analyzes the properties of optimal lotteries under realistic RDU preferences. The last section concludes.

2 Optimal discrete lotteries

We analyze in this section which type of preferences is consistent with profit-maximizing firms offering lotteries with a finite number of payoffs (or discrete lotteries for short). The possibility of continuous lotteries is considered in the next section.

2.1 The model

A lottery consists of n payoffs $(x_i)_{i=1,\dots,n}$, and $n+1$ cumulative probabilities $(\pi_i)_{i=0,1,\dots,n}$. Payoffs belong to an interval I (which can be the set of real numbers $\mathbb{R}$, or bounded above or below, with closed or open ends). π_i is the probability that the consumer gets x_i or less (with $\pi_0 = 0$ and $\pi_n = 1$). In a commercial lottery, payoffs are prizes net of the ticket price and the smallest payoff(s) is (are) negative to ensure a positive profit to the firm. The profit of the risk-neutral firm selling the lottery writes:

$$\Pi = -\sum_{i=1}^{n} (\pi_i - \pi_{i-1}) x_i$$

Consumers are RDU decision makers[1]. Both nonlinear weighting of probabilities and nonlinear utility influence risk preferences:

Definition 1 *Denote u a strictly increasing and continuously differentiable function on I. Let g be a strictly increasing and continuously differentiable from $[0,1]$ to itself, satisfying $g(0) = 0$ and $g(1) = 1$. The agent is RDU with utility function u and probability weighting function g if the value U he derives from the lottery writes:*

$$U = \sum_{i=1}^{n} \left(g(\pi_i) - g(\pi_{i-1}) \right) u(x_i)$$

Instead of analyzing a monopoly firm maximizing profit under the participation constraint of the consumer, we look at the dual problem of maximizing the player's utility subject to a minimum profit for the firm. This is done for convenience reason, as it is strictly equivalent. In both cases we are looking for a Pareto optimum.[2]

The lottery $\{x_i, \pi_i; i = 1, ..., n\}$ is optimal if player's utility is maximized under the constraints that the firm obtains at least a profit equal to B, and that π_i and x_i are increasing:

[1]RDU model is a simple and powerful generalization of the expected utility model. It is able to explain the behavior observed in the Allais paradox, as well as for the observation that many people both purchase lottery tickets and insure against losses. See Machina (1994) and Diecidue and Wakker (2001) for an introduction to RDU theory.

[2]In every allocation problem with non-satiated preferences, when searching for a Pareto equilibrium, it is equivalent to set a minimum utility for the first agent, and maximize the utility of the second agent, or to do the opposite. In particular, it does not matter much how the surplus from trade is split between the firm and the consumer.

$$\max_{n}\begin{cases} \max \sum_{i=1}^{n} [g(\pi_i) - g(\pi_{i-1})]u(x_i) \\ \text{s.t.} \quad -\sum_{i=1}^{n} (\pi_i - \pi_{i-1})x_i = B \\ x_{i+1} - x_i \geq 0,\ i = 1, ..., n-1 \\ \pi_i - \pi_{i-1} \geq 0,\ i = 1, ..., n \\ \pi_0 = 0,\ \pi_n = 1 \end{cases}$$

Remark 1 *Payoffs and cumulative probabilities must be increasing sequences so as to satisfy RDU preferences and definition of cumulative probabilities. Every time one of the two ordering constraints binds, the number of distinct payoffs included in the lottery is reduced by one. This is obviously the case if* $x_{i+1} = x_i$*, where the two payoffs have the same probability equal to* $\pi_{i+1} - \pi_{i-1}$*. If* $\pi_i = \pi_{i-1}$ *the probability of winning* x_i *is simply zero. In both cases, removing the i-th prize does not change the nature of the lottery.*

Hence we can run the maximization problem with a large number of payoffs, and end up with a more limited number of effective prizes satisfying both $\pi_i > \pi_{i-1}$ and $x_{i+1} > x_i$.

This is a two step maximization: n payoffs and probabilities are selected, but n is also optimally chosen. Therefore an optimal lottery will have to satisfy two sets of conditions. Let us write the two problems separately:

$$v_n = \begin{cases} \max \sum_{i=1}^{n} [g(\pi_i) - g(\pi_{i-1})]u(x_i) \\ \text{s.t.} \quad -\sum_{i=1}^{n} (\pi_i - \pi_{i-1})x_i = B \\ x_{i+1} - x_i \geq 0,\ i = 1, ..., n-1 \\ \pi_i - \pi_{i-1} \geq 0,\ i = 1, ..., n \\ \pi_0 = 0,\ \pi_n = 1 \end{cases} \tag{1}$$

and

$$V = \max_{n} v_n \tag{2}$$

Solving for v_n is a relatively classical problem. n-optimal lotteries are lotteries that solve this problem with a fixed number of prizes. They will be analyzed in Subsection 2.3. Solving for V is a bit different. From Remark 1, we can always add superfluous prizes to an existing lottery. Starting from a lottery with n prizes, we can replicate its outcome with $m > n$ prizes, so that increasing the number of prizes cannot worsen the outcome. But if a lottery with n prizes is optimal, adding new prizes will not improve the outcome. We analyze implied restrictions on lotteries in Subsection 2.2.

2.2 Optimality conditions

A Kuhn-Tucker function is formed in which the multipliers for the profit constraint, the ordering constraints $x_{i+1} - x_i \geq 0$, $i = 1, ..., n-1$ and $\pi_i - \pi_{i-1} \geq 0$, $i = 1, ..., n$, are respectively λ, μ_i and ν_i. Consistent with remark 1, a payoff x_i is selected in the lottery if $\mu_i = 0$ and $\nu_i = 0$. The following proposition shows under which conditions a payoff between two selected payoffs is also selected at the optimum (proofs deferred to Appendix A):

Proposition 1 *We have the following necessary conditions for payoffs* $(x_i, \pi_i)_i = 1..n$ *to be selected:*

$$\frac{g(\pi_i) - g(\pi_{i-1})}{\pi_i - \pi_{i-1}} u'(x_i) = \lambda, \quad i = 1, ..., n \tag{3}$$

$$\frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} g'(\pi_i) = \lambda, \quad i = 1, ..., n-1. \tag{4}$$

$$u''(x_i) \leq 0, \quad i = 1, ..., n \tag{5}$$

$$g''(\pi_i) \geq 0, \quad i = 1, ..., n-1. \tag{6}$$

Eq. (3) shows how probability distortion affects prize fixing. Where the ratio $[g(\pi_i) - g(\pi_{i-1})]/(\pi_i - \pi_{i-1})$ is greater than one, the probability $(\pi_i - \pi_{i-1})$ of winning x_i is overestimated, leading the operator to raise the associated prize. This operation raises players' value of gambling and relaxes the participation constraint. A similar reasoning holds for the choice of cumulative probabilities in Eq. (4). Increasing π_i gives a greater weight to prize x_i and a smaller one to x_{i+1}. This is profitable where the ratio before $g'(\pi_i)$ is high, i.e. where players value much the payoff increment.[3]

Optimality conditions only check that payoffs and probabilities are optimal but do not guarantee that the lottery is itself optimal. We cannot rule out the possibility that an expanded lottery including all initial payoffs plus at least one additional payoff does not raise consumers' utility for a given level of profit. This issue is handled in the next subsection.

2.3 Exclusion conditions

In accordance with Remark 1, we can always create a lottery with $n+1$ payoffs that yields the same outcome as a lottery with n payoffs. Equivalently, the sequence v_n in problem (1) is non decreasing. It will therefore have a supremum as long as it is bounded. But if a discrete lottery L_n with n payoffs is optimal, then the supremum is a maximum. Allowing for more prizes will not improve the outcome. In particular, adding a new prize (x_i, π_i) to an already optimal lottery should lead to $x_i = x_{i+1}$ or $\pi_i = \pi_{i-1}$.

Proposition 2 *(i) Let us consider a lottery with at least two selected payoffs x_{i-1} and x_{i+1} and an intermediate payoff x_i with probability $\pi_i \in]\pi_{i-1}, \pi_{i-1}[$. If the firm finds optimal to merge x_i with x_{i+1} then*

$$\frac{g(\pi_i) - g(\pi_{i-1})}{\pi_i - \pi_{i-1}} \geq \frac{g(\pi_{i+1}) - g(\pi_i)}{\pi_{i+1} - \pi_i} \tag{7}$$

[3]Prize optimality condition (3) is in Quiggin (1991) but not its counterpart (4) on optimal probabilities as he makes the simplifying assumption of a uniform probability distribution.

(ii) Let us assume a lottery with at least two selected payoffs x_{i-1} and x_{i+1} and an intermediate payoff $x_i \in]x_{i-1}, x_{i+1}[$. If $\pi_i = \pi_{i-1}$ (i.e. the probability of x_i is optimally set to zero) then

$$\frac{u(x_i) - u(x_{i-1})}{x_i - x_{i+1}} \leq \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} \tag{8}$$

The proof is deferred to Appendix A. Condition (7) can be rewritten as

$$g(\pi_i) \geq \frac{\pi_{i+1} - \pi_i}{\pi_{i+1} - \pi_{i-1}} g(\pi_{i-1}) + \frac{\pi_i - \pi_{i-1}}{\pi_{i+1} - \pi_{i-1}} g(\pi_{i+1})$$

which has a simple geometric interpretation. The probability weighting function $g(\pi_i)$ of any excluded payoff must be above the line crossing the weighting function of the nearest left and right selected cumulative probabilities. While being locally convex around each selected probability (see Eq. (6)), its slope must decrease between two selected probabilities. Hence the weighting function is concave-convex-concave between two selected probabilities. Figure 1. is an example of such a weighting function.[4]

<Include Figure 1 here>

Likewise, condition (8) can be rewritten as

$$u(x_i) \leq \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}} u(x_{i-1}) + \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} u(x_{i+1})$$

which states that the utility function of all payoffs whose probability is set to zero must be below the line crossing the utility function of the nearest left and right selected payoffs. While locally being locally concave around each selected payoff (5), its slope must increase between two selected payoffs. Figure 2 provides an illustration of such a utility function.

[4]Proposition 1 and 2 are also compatible with curves with more alternations. This would imply even less plausible preferences. The same remark applies to Fig. 2.

<Include Figure 2 here>

To sum up, proposition 1 describes optimal lotteries for which prize values and probabilities are optimally chosen but the number of payoffs is given. The payoff set is next made optimal by checking that the inclusion of new prizes does not change the number of payoffs actually offered. Implied exclusion conditions are given in proposition 2. Overall, those constraints impose implausible restrictions on the utility and weighting functions. An optimal lottery with a finite number of payoffs requires that the utility function has a concave-convex-concave shape between any two consecutive payoffs and that the probability weighting function has a convex-concave-convex shape between any two adjacent cumulative probabilities. This configuration might be defended in the two-payoff case as the utility function resembles the one advocated by Friedman and Savage (1948). But since we need twice as many curvature changes of the utility and weighting functions as the number of payoffs, it becomes harder and harder to justify lotteries with more and more prizes.

We may conclude from this section that with more standard preferences (a concave utility and/or a convex weighting function), adding new payoffs will always improve consumers' utility or firm's profit. More formally the sequence v_n of values for lotteries with n prizes will be strictly increasing. But if v_n is bounded, this sequence will converge, so that the marginal gain of adding one extra prize will decrease with the number of prizes. In this case, although a lottery with a finite number of payoffs is not optimal, it might be a good approximation of an optimal lottery. We need however to better describe optimal lotteries, which is the aim of the next section.

3 Optimal lotteries for rank dependent utility players

The previous section starts from a generic form of lotteries and explores under which type of risk preferences they are optimal. The method is reversed here. We assume that players are characterized by realistic risk preferences and see which type of lotteries they prefer. General results which hold for a large class of utility and weighting functions are first presented. We then focus on optimal prize distributions under more specified risk preferences.

Only distributions with a finite number of payoffs have been considered so far. All types of cumulative distribution function (Cdf) F, with weak derivative f are now allowed, including probability mass functions[5]. The only restriction we impose is that payoffs are selected within a closed bounded interval $[a;b]$ that may be arbitrarily large. The continuous equivalent of the discrete RDU problem writes:

$$\begin{cases} \max U(F) = \int_a^b f(x) g'(F(x)) u(x) dx \\ \text{s.t. } \Pi(F) = -\int_a^b f(x) x dx = B \\ F \text{ is a Cdf on } [a;b] \end{cases} \tag{9}$$

where U and Π are the consumer's payoff and firm's profit. A lottery is optimal if consumer's utility is maximum under the firm's participation constraint.

Appendix B exhibits conditions for existence, uniqueness, constructibility and continuity of an optimal lottery. In particular, conditions are found under which the optimal Cdf will be continuous and strictly increasing (neither vertical nor flat).

[5]Weak derivatives extends the notion of derivatives for non differential functions, by using integration by part. f is the weak derivative of F if for any differentiable function G with derivative g and any (a, b), we have $\int_a^b fG = [F(b)G(b) - F(a)G(a)] - \int_a^b Fg$.

3.1 Realistic risk preferences

We are now able to characterize the payoff distribution properties when empirically founded utility and weighting functions are assumed. Experimental literature has consistently found an inverse-S shaped probability weighting function (Camerer and Ho, 1994; Bleichodt and Pinto, 2000; Etchart-Vincent, 2004) and a concave utility function in the gain domain (Tversky and Kahneman, 1992; Wu and Gonzalez, 1996; Abdellaoui, 2000, and Abdellaoui, Bleichrodt, and Paraschiv, 2004). There is less agreement about the shape of the utility function over losses. Abdealloui et al. (2008) review the evidence and conclude that the utility function is slightly convex but close to linearity. We assume that the utility function is also concave on losses as a convex part would entail a demand for a large (even infinite) negative payoff.

Definition 2 *A weighting function is inverse-S shaped with inflection point δ if it is strictly concave over $[0;\delta[$ and strictly convex over $]\delta;1]$.*

Proposition 3 *Suppose that u is strictly concave over [a;b], and that g is inverse S-shaped with inflection point δ. Then:*

(i) the smallest payoff $x_0 \geq a$ has a discrete probability $\pi_0 \geq \delta$, with the slackness condition $(x_0 - a)(\pi_0 - \delta) = 0$,

(ii) The maximum payoff $x_1 \leq b$ has a discrete probability $1 - \pi_1 \geq 0$, with the slackness condition $(b - x_1)(1 - \pi_1) = 0$,

(iii) the probability distribution is continuous over $]x_0;x_1[$, and the Cdf F is characterized by

$$g'(F(x))u'(x) = \lambda \qquad x \in]x_0;x_1[\tag{10}$$

with λ a constant.

The proof is deferred to Appendix B. A discrete probability on a and b means a constrained optimum in which the interval constraint binds. If we were to run the problem on a bigger interval $[\tilde{a}, \tilde{b}] \supseteq [a, b]$, a and b would become interior points, and we could use the smoothing argument of Property 4 in Appendix B.

If the problem is well behaved, we would hope that, when extending the interval $[a, b]$, this constraint will stop binding at one point: the support of the distribution stops increasing with the interval. Or, at least, that the minimum prize (the ticket price) remains unchanged, and the discrete probability of the maximum payoff converges to zero. This is likely to be true if $\lim_{x\to\infty} u'(x) = 0$. In this case, if $g'(1)$ is finite, there will be a point where equation (10) will no longer hold, and the support of the distribution will end there. The marginal utility becomes so low, that, even with weighted probabilities, it makes no sense to make those events possible. If we have $\lim_{x\to\infty} u'(x) = 0$ and $\lim_{t\to 1} g'(t) = +\infty$, we would probably have that $\lim_{x\to\infty} F(x) = 1$, so that there is no discrete probability on $+\infty$. The marginal utility becomes so low that the marginal weight needs to be ever higher to compensate, so that the cumulative probability converges to 1[6].

An interesting feature of any optimal lottery is that the minimal prize has a positive probability mass, which is reminiscent of actual lotteries in which the ticket price is lost with a positive probabilities and no prizes below it are included in the lottery. Another remarkable property is that, provided the upper bound constraint is not binding, prizes above the minimum payoff are continuously distributed. Over the continuous interval, prizes and probabilities are determined by the optimality condition (10). It is the continuous analog of proposition (1). As larger prizes are included in the lottery, the disutility

[6]The case studied below of a power-law distribution provides an example of an optimal distribution with unbounded support.

of spreading prizes (recall u is concave) must be compensated by increasing optimism about their realization, which is made possible by the convexity of g. Condition (10) can also be differentiated with respect to x:

$$f(x) = \frac{-u''(x)}{u'(x)} \frac{g'(F(x))}{g''(F(x))}$$

The lower the curvature of u or the higher the convexity of g, the lower the probability density f and the more spread out the prize distribution.

3.2 Power-law distribution

A comprehensive theory of optimal lottery should explain why commercial lotteries are characterized by highly skewed prize distributions, in which huge amounts of money are offered with close-to-zero probabilities. Such a property naturally emerges under simple specifications.

Definition 3 *The utility function is* $u(x) = x^\sigma$ *with* $0 < \sigma < 1$ *over the gain domain* $[0, +\infty]$*; the weighting function has a constant relative sensitivity (CRS):*

$$g(\pi) = \begin{cases} \delta^{1-\gamma}\pi^\gamma & if\ 0 \leq \pi \leq \delta \\ 1 - (1-\delta)^{1-\gamma}(1-\pi)^\gamma & if\ \delta < \pi \leq 1 \end{cases} \tag{11}$$

for $0 < \delta < 1$ *and* $0 < \gamma < 1$*.*

The utility function is a standard choice in most empirical works on risk attitude. CRS functions are the equivalent of power utility function for weighting functions. They belong to the class of switch-power weighting functions axiomatized by Diecidue et al. (2009) and have been investigated by Abdellaoui et al. (2010). A key property is their inverse S-shape which implies diminishing sensitivity, that is, sensitivity to changes in probability decreases as probability moves away from the reference points 0 and 1. Other functional forms are possible (e.g. Tversky and Kahneman (1992) or Lattimore et al. (1992)). All share this central property. We are interested in the extent to which consumers distort cumulative probabilities close to one, i.e. where $\pi > \delta$. Accordingly, let us pose the dual function

$$h(1-\pi) = 1 - g(\pi) = (1-\delta)^{1-\gamma}(1-\pi)^{\gamma}$$

over the right-hand part of the distribution. The index of relative curvature of this function is:

$$-\frac{(1-\pi)h''(1-\pi)}{h'(1-\pi)} = 1-\gamma,$$

hence the name of constant relative sensitivity. It allows a separation of two sources of probability distortion: the degree of relative optimism measured by the elevation parameter δ (see Abdellaoui et al., 2010) and the degree of sensitivity towards probabilities of extreme outcomes measured by the parameter γ. The smaller γ, the more concave the dual weighting function, and the more sensitive are consumers to changes in cumulative probabilities close to 1. With those assumptions, if we use Eq. (10), even though the interval is not bounded, and $\lim_{t\to 1} g'(t) = +\infty$, what do we get (proof deferred to Appendix B)?

Proposition 4 *Under Definition* 3*, the optimal Cdf is a power-law distribution, with a tail index* $\alpha = (1-\sigma)/(1-\gamma)$.

Prop. 4 gives a simple formula for the tail index of the prize distribution. It puts forth two key psychological factors. The smaller σ, the more concave the utility function and the more costly the spread of prize payouts over the gain domain. The upper tail of the prize distribution is consequently thinner, as a higher tail index diminishes the proportion of extreme prizes. The closer γ to 0, the more sensitive consumers are to changes in cumulative probabilities close to 1 and the greater the overweighting of small probabilities associated with extreme prizes. For a given utility curvature, this leads the firm to skew the prize distribution further so as to make the game more attractive to players who are overoptimistic about the occurrence of extremely unlikely events.

3.3 Tail-fatness estimation

It is interesting to look at whether the optimality condition $\alpha = (1-\sigma)/(1-\gamma)$ found in Prop. (4) is roughly consistent with experimental values and observed tail-fatness of lotteries. Empirical estimates of behavioral parameters σ and γ can be found in the experimental literature. Abdellaoui et al. (2010) find a relative sensitivity coefficient γ around 0.5. Estimates of the power coefficient σ are in the interval $[0.3, 0.9]$ [7] Replaced in the optimality condition, this implies a tail index α between 0.2 and 1.4. All values imply extremely large degrees of tail-fatness. As a reference point, Atkinson and Piketty (2007) report a tail index between 1.5 and 3 for income and Kleiber and Kotz (2003) around 1.5 for wealth.

We may estimate the parameter α for a popular lotto game called Euromillions, launched in 2004 in nine European countries. We choose this game for its wide popularity, prize data availability, and the richness of its prize structure. In this lotto game, five numbers are drawn in the set 1 to 50 and two bonus numbers in the set 1 to 9. There are 12 winning ranks according to how many numbers are guessed in the two sets[8]. Fig. 3 plots $-log(1-F(x))$ over $log(x)$ based on the observation of 378 consecutive drawings

[7] Tversky and Kahneman (1992), Abdellaoui (2000) and Abdellaoui et al. (2007) find resp. $\sigma = 0.88$, 0.89 and 0.75 while Wu and Gonzalez (1996), Camerer and Ho (1994) and Fennema and van Assen (1998) find lower values (resp. $\sigma = 0.52$, 0.37 and around 0.3).

[8] See Roger (2011) for a more detailed description of the game.

run between February-2004 and May 2011[9].

<Include Figure 3 around here>

Each point represents one of the 3,616 prize payouts recorded in the data. Assuming that players face a time-invariant prize distribution[10], we can see that prizes (net of the ticket cost) extend across a very large interval from 5 to $129,818,429$ euros with few gaps in between. Hence a player may hope to win almost any level of gains with a positive probability over an extremely large prize set. This property comes from the parimutuel nature of the game in which money prizes greatly vary across drawings. Total prizes are set equal to a percentage of the total amount bet[11]. As ticket sales and the number of winners at each rank fluctuates across draws, prize values are random. The more tickets sold, the larger the pie to be divided among winners. The greater the number of winners at a given rank, the smaller the individual shares at this rank. These two factors are random and explain why the prize distribution spans a very large prize space.

Figure 3 fits a linear regression model between the log countercumulative distribution function and log prize. The relationship is close to linearity despite local fluctuations around the trend, with a R^2 of 0.94. The average slope, an estimate of the tail index α[12] is 1.066 which denotes a highly skewed prize distribution. It is within the range [0.2, 1.4] found in the literature previously described. Hence, the optimality condition in Prop. 4 is at first look consistent with plausible values of the preference parameters. Another

[9] Data on payouts and numbers of winners at each rank is available on the French lottery operator web page www.francaise-des-jeux.fr.

[10] This is not always verified in practice due to episodes in which jackpots are rolled over in the absence of winners. Suppressing rollovers from the data set would only affect the distribution of extreme gains.

[11] A winner at rank $i = 1, ..., 12$ receives an amount equal to $\varphi_i(1-\tau)S/x_i$ where φ_i is the share of (net of tax τ) proceeds S dedicated to rank i winners, and x_i the number of winners at rank i.

[12] The log countercumulative distribution function and log prize are linearly related in power-law distributions.

implication is that while diminishing marginal utility is a necessary condition for a prize distribution to be optimal, this does not prevent the marketing of highly skewed lotteries.

4 Conclusion

This article considers lotteries as Pareto equilibria between a firm and RDU consumers. We find that lotteries with a discrete number of prizes are incompatible with standard RDU preferences, but continuous lotteries are. The minimal prize has a positive probability mass and prizes above it are continuously distributed. Under realistic utility and weighting functions the optimal prize structure follows a power-law distribution. A back-of-the-envelope calibration based on data from the lotto game Euromillions suggests that an extreme value of tail-fatness is still compatible with a realistic degree of concavity of the utility function and convexity of the weighting function.

Our conclusion that RDU models cannot explain lotteries with a discrete number of payoffs should be treated with caution for at least two reasons. First, while lotteries with a discrete number of prizes cannot be optimal, the profit and utility that they bring might be close to the optimum reached with a continuum of prizes. This is a quantitative issue that only a calibration exercise could settle. Second, commercial lotteries have a finite number of prizes because of practicality and simplicity. Here our model misses possibly important physical constraints independent of our preference assumptions. On the other hand, it is also possible that even lotteries with a very large number of prizes, a proxy for continuous lotteries, could be dominated by lotteries with fewer prizes for behavioral reasons not explained by RDU preferences. This would be the case if individuals feel uneasy with complex random processes or distrust lotteries with many prizes as being less transparent. Those hypotheses outline the necessity to pursue additional research to narrow the gap between prize structures observed in commercial lotteries and risk theory.

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Appendix A Proofs for discrete lotteries

Proof of Proposition 1

First order conditions (FOC) with regard to payoff x_i are:

$$(g(\pi_i) - g(\pi_{i-1}))u'(x_i) - \lambda(\pi_i - \pi_{i-1}) + \mu_{i-1} - \mu_i = 0 \tag{12}$$

with $\mu_i = \mu_{i-1} = 0$ as x_i and x_{i-1} are also selected payoffs. For x_1 and x_n, there is no corresponding μ_0 nor μ_n, so they do not need to be set to 0. The FOC with regard to π_i is:

$$-(u(x_{i+1}) - u(x_i))g'(\pi_i) + \lambda(x_{i+1} - x_i) + \nu_i - \nu_{i+1} = 0 \tag{13}$$

Since x_i and x_{i+1} are two selected payoffs, $\nu_i = \nu_{i+1} = 0$.

The necessary second order conditions write as the negative semi-definiteness of the Hessian matrix. In particular, it implies that all the diagonal elements $(g(\pi_i) - g(\pi_{i-1}))u''(x_i)$ and $-(u(x_{i+1}) - u(x_i))g''(\pi_i)$ of this matrix are non positive. We conclude with $g(\pi_i) > g(\pi_{i-1})$ and $u(x_{i+1}) > u(x_i)$ as the payoffs are selected.

We assumed without loss of generality that all payoffs were selected. If some were superfluous, we could simply remove them. □

Proof of Proposition 2

(i) Since x_{i-1} and x_{i+1} are selected, $\mu_{i-1} = \mu_{i+1} = 0$. $x_i = x_{i+1}$ implies $\mu_i \geq 0$. The FOC associated with x_i and x_{i+1} are given by Eq. (12):

$$(g(\pi_i) - g(\pi_{i-1}))u'(x_i) - \lambda(\pi_i - \pi_{i-1}) - \mu_i = 0$$

$$(g(\pi_{i+1}) - g(\pi_i))u'(x_{i+1}) - \lambda(\pi_{i+1} - \pi_i) + \mu_i = 0.$$

This implies:

$$\frac{g(\pi_i) - g(\pi_{i-1})}{\pi_i - \pi_{i-1}}u'(x_i) \geq \lambda \geq \frac{g(\pi_{i+1}) - g(\pi_i)}{\pi_{i+1} - \pi_i}u'(x_i) \qquad \square$$

(ii) As x_{i-1} and x_{i+1} are selected $\nu_{i-1} = \nu_{i+1} = 0$. $\pi_i = \pi_{i-1}$ implies $\nu_i \geq 0$. From Eq. (13), the FOC associated with π_{i-1} and π_i are respectively:

$$(u(x_i) - u(x_{i-1}))g'(\pi_i) - \lambda(x_i - x_{i-1}) + \nu_i = 0$$

$$(u(x_{i+1}) - u(x_i))g'(\pi_i) - \lambda(x_{i+1} - x_i) - \nu_i = 0,$$

implying:

$$\frac{u(x_i) - u(x_{i-1})}{x_i - x_{i-1}}g'(\pi_i) \leq \lambda \leq \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i}g'(\pi_i) \qquad \square$$

Appendix B Proofs for continuous lotteries

Properties of continuous lotteries

The Appendix begins by proving four general properties of optimal lotteries: existence (Property 1), uniqueness (Property 2), constructibility (Property 3) and continuity (Property 4). We first need to pose alternative formulations of utility and profit (Remark 2) and recall Helly's selection theorem (Theorem 1). Last, the Appendix provides the proofs of Prop. (3) and (4) which address the optimal shape of lotteries.

Remark 2 *Alternative formulations of preferences and profit. Three alternative formulations are used in the proofs of this Appendix. The second expressions are integrated by part (Eq. (15) and (18)) and in the third ones $F(x)$ is inverted into $x(t)$ (Eq. (16) and (19)). Players' preferences may alternatively be written as:*

$$U(F) = \int_a^b f(x)g'(F(x))u(x)dx \tag{14}$$

$$U(F) = u(b) - \int_a^b g(F(x))u'(x)dx \tag{15}$$

$$U(F) = \int_0^1 g'(t)u(x(t))dt \tag{16}$$

Likewise, the firm's profit expresses as:

$$\Pi(F) = -\int_a^b f(x)xdx = B \tag{17}$$

$$\Pi(F) = -b + \int_a^b F(x)dx \tag{18}$$

$$\Pi(F) = -\int_0^1 x(t)dt \tag{19}$$

To prove the existence of an optimal lottery, we need Helly's selection theorem, a compactness result for increasing functions.

Theorem 1 *Helly's selection theorem: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of increasing functions from $\mathbb{R}$ to $[0;1]$. Then, there exists a subsequence converging pointwise to an increasing function f from $\mathbb{R}$ to $[0;1]$. Furthermore, the number of discontinuities of f is at most countable.*

Property 1 *When the payoff space is closed and bounded, an optimal lottery exists.*

Proof Let us define a lottery as a non-decreasing function F from the interval $[a;b]$ to $[0;1]$. Since $[a;b]$ is bounded, the utility that the consumer can get from a lottery is also bounded. Denote S the set of non-decreasing functions from $[a;b]$ to $[0;1]$ that also satisfy the profit constraint of the firm. U has an upper bound on S. Then there exists a sequence of non-decreasing functions whose utility converges to the upper bound: $\exists (F_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}, U(F_n) \to sup_S U$. According to Helly's selection theorem, there exists a subsequence $(F_{\varphi(n)})_{n\in\mathbb{N}}$ converging pointwise (and therefore in distribution) to a non decreasing function F.

Since the prize space $[a;b]$ is closed and bounded, we can integrate by parts (Remark 2), so with $\int_a^b F(x)dx = \lim_{n\to\infty}\int_a^b F_{\varphi(n)}(x)dx$, we have $\Pi(F) = \lim_{n\to\infty}\Pi(F_{\varphi(n)})$, F still satisfies the profit constraint, $F \in S$. Next, still using Remark 2, since the prize space $[a;b]$ is a closed and bounded interval, the derivative of u is a bounded function, as it is continuous, and therefore

$$\int_a^b g(F_{\varphi(n)}(x))u'(x)dx \overset{n\to\infty}{\to} \int_a^b g(F(x))u'(x)dx$$

so that $U(F) = sup_S(U)$, hence F is optimal. □

Some additional restrictions are needed for uniqueness:

Property 2 *If an optimal lottery exists and if u is strictly concave or g strictly convex, then the lottery is unique.*

Proof Let us denote $x(t)$ the inverse of the cumulative distribution function $F(x)$. If two different lotteries $x_1(t)$ and $x_2(t)$ yield the same optimal utility defined in Eq. (16) and if the function u is concave, we have

$$\forall t, u(\frac{x_1(t)+x_2(t)}{2}) \geq \frac{u(x_1(t))+u(x_2(t))}{2},$$

the inequality being strict whenever u is strictly concave and $x_1(t) \neq x_2(t)$. Therefore, if we write F the function associated with $x(t) = \frac{x_1(t)+x_2(t)}{2}$, if u is strictly concave and x_1 and x_2 are not almost surely equal we have $U(F) > \frac{U(F_1)+U(F_2)}{2} = sup_S U$, because we integrate a strict difference. This is contradictory. Hence the optimal lottery is unique. The same reasoning applies with the second expression of consumer's utility (15): whenever g is strictly convex, $-g$ is strictly concave, and the same computations bring uniqueness. □

We have found conditions of existence and uniqueness of an optimal lottery, but we do not know what it looks like, or how to find it. The next property tells us that it can be seen as the limit of continuous functions, or of step functions. It also tells us that if we run the maximization program for 2-payoff lotteries, 3-payoff lotteries, and so on, we will converge to the optimal lottery.

Property 3 *(i) An optimal lottery can be expressed as the pointwise limit of continuous functions (ii) An optimal lottery can be expressed as the pointwise limit of right-continuous step functions (discrete lotteries) (iii) If the optimal lottery is unique, it can be expressed as the pointwise limit of n-optimal lotteries.*

The proof comes from the fact that under certain conditions, step functions can be approximated by continuous functions, and continuous functions by step functions. This proposition will be useful as some desirable properties of optimal lotteries will be retained when taking the limit.

Proof (i) and (ii): Write the function as the sum of a continuous function and a discontinuous step function. From Helly's theorem, the number of discontinuities is at most countable. First, the discontinuous function can be approximated pointwise by a sequence of continuous functions, and by a sequence of right-continuous step functions. Second, since we are on a closed bounded interval, the continuous function can be approximated pointwise by a sequence of continuous functions and by a sequence of step functions. Therefore, an optimal lottery can be written as the pointwise limit of continuous lotteries, and also as the limit of discrete lotteries. (iii) If $(F_n)_{n\in\mathbb{N}}$ is a sequence of lotteries converging to F, where F_n has at most n prizes, write H_n a n-optimal lottery. $U(F_n) \leq U(H_n) = sup_S(U) = \lim_{k\to\infty} U(F_k)$, so that $\lim_{n\to\infty} U(H_n) = sup_S(U)$. With Helly's theorem, a subsequence of (H_n) converges to some H. And because of unicity of the optimal lottery, $H = F$, any optimal lottery can be written as the limit of n-optimal lotteries. □

We now turn to the important issue of continuity of the distribution. In the case of discrete lotteries (see (5) and (6)), the utility function cannot be strictly convex at interior payoffs (different from a or b, the bounds of the payoff interval). Likewise, the weighting function cannot be strictly concave at interior cumulative probabilities (different from 0 or 1). This means that for n-optimal lotteries, the Cdf will be constant (no positive probability) on any interval where the utility function is strictly convex, and the Cdf will jump when the weighting function is strictly concave (no selected probability). This is still true for optimal continuous distributions, since it is the pointwise limit of n-optimal lotteries (Property 3). Therefore, the Cdf must jump wherever the weighting function is strictly concave, and it must stay constant wherever the utility is strictly convex. Is it possible that the Cdf jumps even though the weighting function is strictly convex, or that it stays constant even though the utility is strictly concave? The answer is negative:

Property 4 *If the weighting function g is strictly convex on $[c;d] \subset [0;1]$, then the optimal cumulative distribution function F cannot jump from c to d at an interior point. If the utility u is strictly concave on $[x;y] \subset [a;b]$, then F cannot be constant on $[x;y]$.*

The basic idea of the proof is that it is first-order optimal to "smooth" the vertical and horizontal parts of the Cdf. Property 4 shows that at interior points where the weighting function is convex and the utility function is concave, the Cdf is strictly increasing, so that it is continuous. In other words, the payoff probability distribution is continuous wherever the utility function is concave and the weighting function convex (and not continuous otherwise, as already stated previously).

Proof The steps of the proof are the following: we take the Cdf with the jump, transform the jump into a steep, yet continuous function, and show that the transformation brings a marginal gain. Suppose that F does jump from $c = F(x_0^-)$ to $d = F(x_0^+)$, at an interior point x_0. Write $H(y) = 0$ if $y < -1/2$, $H(y) = \frac{1}{2} + y$ if $y \in \left[-\frac{1}{2}; \frac{1}{2}\right]$, and 1 otherwise. Write $h(t,x) = (d-c)H(\frac{x-x_0}{t})$. Then write $F(x) = C(x) + h(0,x)$, where the function C is continuous in x_0, and $h(0,x)$ is the pointwise limit of h when $t \to 0^+$, and write $F(t,x) = C(x) + h(t,x)$, so that $F(x) = F(0,x)$. It is obvious that the profit of the firm does not depend on t. Now, write $U(t) = U(F(t,x))$, the utility of the consumer defined in Eq. (15). We have:

$$U(t) - U(0) = -\int_{x_0 - t/2}^{x_0 + t/2} (g(F(t,x)) - g(F(0,x)))u'(x)dx$$

Writing $x = x_0 + ty$,

$$U(t) - U(0) = \int_{-1/2}^{1/2} (g(F(0,x_0+ty)) - g(F(t,x_0+ty)))u'(x_0+ty)(tdy)$$

$$U(t) - U(0) = t.\int_{-1/2}^{1/2} \begin{bmatrix} g(C(x_0+ty) + h(0,x_0+ty)) \\ -g(C(x_0+ty) + h(t,x_0+ty)) \end{bmatrix} u'(x_0+ty)dy$$

$$\begin{aligned} U(t) - U(0) &= t.\int_{-1/2}^{0} \begin{bmatrix} g(C(x_0+ty)) \\ -g(C(x_0+ty) + (d-c)H(y)) \end{bmatrix} u'(x_0+ty)dy \\ &\quad +t.\int_{0}^{1/2} \begin{bmatrix} g(C(x_0+ty) + (d-c)) \\ g(C(x_0+ty) + (d-c)H(y)) \end{bmatrix} u'(x_0+ty)dy \end{aligned}$$

$$U(t)-U(0) = t.\int_{-1/2}^{0}\begin{bmatrix} g(C(x_0+ty)) \\ -g(C(x_0+ty)+(d-c)(\frac{1}{2}+y)) \end{bmatrix} u'(x_0+ty)dy$$
$$+t.\int_{0}^{1/2}\begin{bmatrix} g(C(x_0+ty)+(d-c)) \\ -g(C(x_0+ty)+(d-c)(\frac{1}{2}+y)) \end{bmatrix} u'(x_0+ty)dy$$

Therefore

$$U'(0) = \lim_{t\to 0^+}\frac{U(t)-U(0)}{t}$$
$$U'(0) = \int_{-1/2}^{0}(g(C(x_0)) - g(C(x_0)+(d-c)(\frac{1}{2}+y)))u'(x_0)dy$$
$$+\int_{0}^{1/2}(g(C(x_0)+(d-c)) - g(C(x_0)+(d-c)(\frac{1}{2}+y)))u'(x_0)dy$$
$$U'(0) = \int_{-1/2}^{0}(g(c) - g(c+(d-c)(\frac{1}{2}+y)))u'(x_0)dy$$
$$+\int_{0}^{1/2}(g(d) - g(c+(d-c)(\frac{1}{2}+y)))u'(x_0)dy$$

If we write, for matters of convenience $G(z) = g(c+(d-c)(\frac{1}{2}+y))$ then $G(-\frac{1}{2}) = g(c)$, $G(\frac{1}{2}) = g(d)$, and

$$U'(0) = \int_{0}^{1/2}((G(\frac{1}{2}) - G(y)) - (G(-y) - G(-\frac{1}{2})))u'(x_0)dy$$

Since g is strictly convex, so is G, and therefore for every $y \in [0;\frac{1}{2}[$, $G(\frac{1}{2}) - G(y) > G(-y) - G(-\frac{1}{2})$, so that $U'(0) > 0$. This means that the cumulative distribution function was not optimal, and a small "smoothing" can increase the profit.

Here, we used expression (15) for consumer's utility. The same can be done with expression (16). The constant parts of the Cdf corresponds to jumps for its inverse. When u is strictly concave, $-u$ is strictly convex, so that the same computations can be used to "smooth" the jumps of the inverse of the Cdf, and therefore, the horizontal parts of the Cdf, as long as interior conditions are satisfied.

Proof of Proposition 3

(i): Proposition 2 shows that for discrete lotteries, optimality implies that the Cdf jumps when the weighting function is concave. Through Property 3, this property is kept for continuous optimal lotteries. Hence the first probability π_0 is at least δ. Suppose now that both $\pi_0 > \delta$ and $x_0 > a$. This means that the Cdf jumps from 0 to π_0 at x_0. Or equivalently the Cdf jumps from 0 to δ, and from δ to π_0 at $x_0 > a$. Since $[\delta, \pi_0]$ is in the convex part of the weighting function, and x_0 is in the interior of the concave part of the utility, this is impossible because of Property 4 in the same appendix. This brings the slackness condition.

(ii): Similarly, at least the payoff x_1 or the probability $1 - \pi_1$ must not be interior, because of Property 4.

(iii): Property 4 shows that when the utility is strictly concave, or when the weighting function is strictly convex, f is strictly positive, so that the increasing constraint on F is not binding. We can integrate by parts the utility and the profit:

$$\begin{aligned} U &= g(\pi_0)u(x_0) + \int_{x_0}^{x_1} f(x)g'(F(x))u(x)dx + (1 - g(\pi_1))u(x_1) \\ &= u(x_1) - \int_{x_0}^{x_1} g(F(x))u'(x)dx \\ \Pi &= -\pi_0 x_0 - \int_{x_0}^{x_1} f(x)xdx - (1 - \pi_1)x_1 = -x_1 + \int_{x_0}^{x_1} F(x)dx \end{aligned}$$

Then problem (9) can be expressed as a Lagrangian, with only F, and not f:

$$L = u(x_1) - \lambda(x_1 + B) - \int_{x_0}^{x_1} \left[g(F(x))u'(x) - \lambda F(x)\right] dx$$

The first order condition with respect to $F(x)$ yields Eq. (10). □

Proof of Proposition 4

The first order optimality condition (10) over the concave part of h (or the right-hand convex part of g) is:

$$h'(1 - F(x))u'(x) = \lambda.$$

Let us differentiate both sides with respect to x:

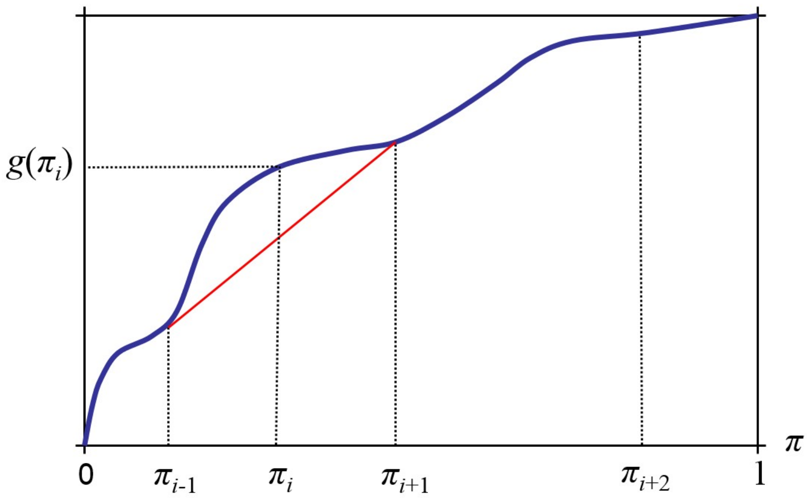
$$-h"(1 - F(x))F'(x)u'(x) + h'(1 - F(x))u"(x) = 0.$$

After a few arrangements, the elasticity of the countercumulative distribution function with respect to payoffs is constant:

$$\varepsilon_{(1-F(x))/x} = -\frac{-xu"(x)}{u'(x)} \frac{h'(1 - F(x))}{-(1 - F(x))h"(1 - F(x))} = -\frac{1 - \sigma}{1 - \gamma}$$

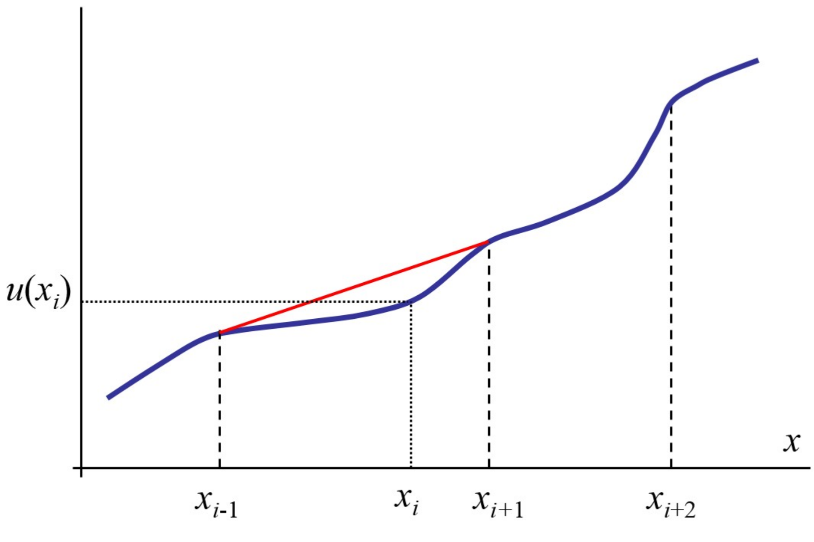
hence F follows a power-law distribution with a tail index $\alpha = (1 - \sigma)/(1 - \gamma)$. □

Figure 1: Example of probability weighting function



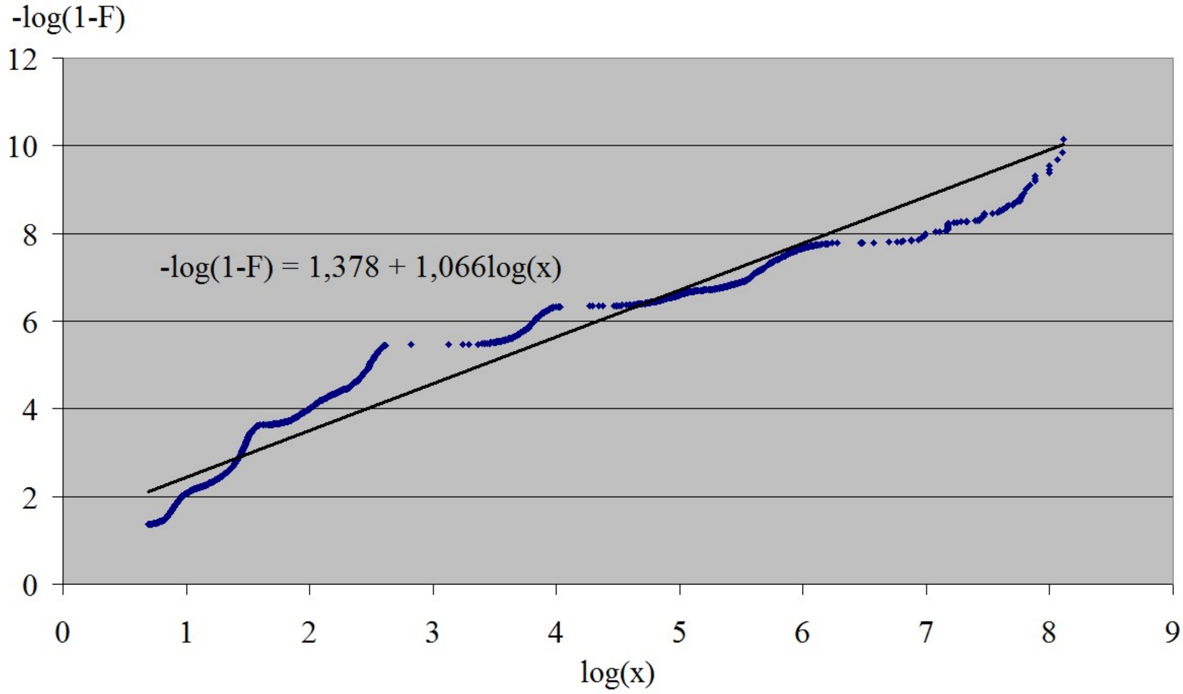
Note: The figure displays a probability weighting function consistent with an optimal lottery endowed with a finite number of prizes. Cumulative probabilities π_{i-1}, π_{i+1} and π_{i+2} are selected probabilities and lie where the weighting function is locally convex. π_i is an example of excluded probability as $g(\pi_i)$ is above the line crossing the weighting function of the nearest left and right selected cumulative probabilities.

Figure 2: Example of utility function



Note: The figure displays a utility function consistent with an optimal lottery endowed with a finite number of prizes. Payoffs x_{i-1}, x_{i+1} and x_{i+2} are selected payoffs. They lie where the utility function is locally concave. x_i is excluded from the lottery as $u(x_i)$ is below the line crossing the utility function of the nearest left and right selected payoffs.

Figure 3: Prize distribution of Euromillions



Notes: $log(1-F(x))$ is log countercumulative distribution function, $log(x)$ is log prize. The computation of the prize distribution is based on the observation of 378 consecutive drawings run between February 2004 and May 2011. The data set includes 3,616 prize payouts and 611,269,599 winners. Source: www.fdj.fr.